AN ALGEBRA-LEVEL VERSION OF A LINK-POLYNOMIAL IDENTITY OF LICKORISH

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Abstract. We establish isomorphisms between certain specializations of BMW algebras and the symmetric squares of Temperley-Lieb algebras. These isomorphisms imply a link-polynomial identity due to W. B. R. Lickorish. As an application, we compute the closed images of the irreducible braid group representations factoring over these specialized BMW algebras.

1. Introduction

In [Li], W. B. R. Lickorish proved the following relation between values of the Kauffman and Jones polynomials of an oriented link:

$$F_L(q^3, q^{-1}+q) = (-1)^{c(L)-1} V_L(-q^{-2}). \tag{1.1}$$

This identity turns out to be a manifestation of a broader phenomenon. There exist two families of finite dimensional algebras (actually von Neumann algebras): on the one hand, Birman-Murakami-Wenzl algebras with a relation between the two parameters suggested by (1.1), and on the other, symmetric squares of Temperley-Lieb algebras. On each side we have a natural trace and a natural homomorphism from the group algebra of a braid group. We show that there is a natural isomorphism between corresponding algebras which respects both structures and therefore "explains" (1.1). The equality of dimensions gives a new combinatorial identity which can be expressed as an explicit bijection between "oscillating" Young tableaux and pairs of ordinary tableaux. Interestingly, our proof of the algebra isomorphism depends on first establishing the combinatorial result; this allows us to show that the natural homomorphism is actually an isomorphism.

The original motivation for this paper was our attempt to understand the closed images of braid groups in the (projective) unitary representations associated with the Kauffman polynomial at $q = e^{\pi i/\ell}$. (For the HOMFLY polynomial, this was done by the first-named author together with M. Freedman and Z. Wang [FLW].) A preliminary

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analysis of the Kauffman polynomial case was undertaken by both authors and Wang [LRW]. The image of any half-twist has eigenvalues q, $-q^{-1}$, and r^{-1}, and [LRW] excludes the cases where the ratio of two eigenvalues is -1, or where all three eigenvalues lie in geometric progression. Certain cases (for example $r = q^{-1}$) degenerate to those considered in [FLW], and the cases excluded in [LRW] that remain are $r = q$, $r = \pm i$, and $r = q^3$. The first set of exceptions will be discussed in the doctoral dissertation of Jennifer Franko; in these cases, the image groups are finite. The second set of exceptions appears to be connected with self-dualities. This paper arose from our discovery that the third set of exceptions had a clean algebraic interpretation. The actual classification of closed images is given in Theorem 7.4 in the final section of the paper.

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2. Combinatorial Notation and Results

The combinatorial language of Young diagrams plays a major role in what follows so we establish notation and terminology for later use.

A Young diagram λ is an array of boxes so that the number of boxes in each row (resp. column) decreases weakly as one reads downwards (resp. to the right), and we denote the set of Young diagrams by YD. Denote by λ_i (resp. $\tilde{\lambda}_i$) the number of boxes in the ith row (resp. column) of λ. We identify λ with an ordered list of its rows $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ or columns $\lambda := [\tilde{\lambda}_1, \ldots, \tilde{\lambda}_j]^t$. The size $|\lambda|$ is defined to be the total number of boxes $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k = \tilde{\lambda}_1 + \cdots \tilde{\lambda}_j$. If $\lambda_i \leq \mu_i$ for all i (where $\lambda_i = 0$ is permitted) we write $\lambda \subset \mu$, and if in addition μ can be obtained from λ by adding one box, we write $\lambda \to \mu$. The relation $\subset$ is encoded in *Young's lattice*. An increasing path in Young's lattice from $[0]$ to λ

$$t_\lambda : [0] = \lambda^{(0)} \to \lambda^{(1)} \to \cdots \to \lambda^{(m)} = \lambda$$

where $|\lambda^{(j)}| = j$ is called a *Young tableau of shape* λ. Denote by $\mathcal{T}(\lambda)$ the set of Young tableaux of shape λ. We shall be particularly interested in Young tableaux t_λ whose shapes $\lambda^{(j)}$ are restricted to a subset of YD. In particular we define a set

$$\Lambda(j, \ell) := \{[j - p, p], 0 \leq j - 2p \leq \ell - 2\}$$

consisting of Young diagrams of size j with at most 2 rows whose row-difference is bounded by $\ell - 2$. This definition makes sense for $3 \leq \ell \leq \infty$, where the case $\ell = \infty$ corresponds to the set of *all* Young

diagrams of size j with at most 2 rows. Notice that $\lambda \in \Lambda(j,\ell)$ is completely determined by its first row, λ_1, since $\lambda_2 = j - \lambda_1$. We set $\Lambda(\ell) = \bigcup_{0\le j} \Lambda(j,\ell)$ so that $\Lambda(\infty)$ is the subset of YD consisting of all diagrams with at most 2 rows. Then we denote by $\mathcal{T}_\ell(\lambda)$ the set of all restricted Young tableaux t_λ where each $\lambda^{(j)} \in \Lambda(j,\ell)$. Observe that $\mathcal{T}_\infty(\lambda) = \mathcal{T}(\lambda)$ since any Young tableaux terminating at a diagram $\lambda \in \Lambda(\infty)$ can only pass through diagrams in $\Lambda(\infty)$.

These notions can be generalized: if $\lambda \to \mu$ *or* $\mu \to \lambda$, *i.e.* λ and μ differ by one box, we write $\lambda \leftrightarrow \mu$. A general path of length m from $[0]$ to λ in Young's lattice

$$o_\lambda : [0] = \lambda^{(0)} \leftrightarrow \lambda^{(1)} \leftrightarrow \cdots \leftrightarrow \lambda^{(m)} = \lambda$$

is called an *oscillating tableau of length* m *and shape* λ. Observe that $j - |\lambda^{(j)}|$ is always a non-negative even number. We denote by $\mathcal{O}(m,\lambda)$ the set of oscillating tableaux of length m and shape λ. We will often restrict the shapes to a subset of YD, in this case the set:

$$\Gamma(\ell) := \{\lambda \in YD : \tilde{\lambda}_1 + \tilde{\lambda}_2 \le 4, \lambda_1 + \lambda_2 \le \ell - 2\} \cup \{[\ell-2, 1^2]\},$$

where we will be interested in the (non-degenerate) cases: $6 \le \ell \le \infty$. For $\ell = \infty$ the conditions reduce to $\tilde{\lambda}_1 + \tilde{\lambda}_2 \le 4$. We denote by $\mathcal{O}_\ell(m,\lambda)$ the set of oscillating tableaux of length m and shape λ restricted to the set $\Gamma(\ell)$.

Our basic combinatorial result is:

Theorem 2.1. *Let* $\lambda, \mu \in \Lambda(m,\ell)$*, and define* $\nu_1 = \lambda_1 + \mu_1 - m$ *and* $\nu_2 = |\lambda_1 - \mu_1|$*. Then if* $\lambda \ne \mu$ *we have:*

$$|\mathcal{O}_\ell(m,[\nu_1,\nu_2])| = |\mathcal{T}_\ell(\lambda)| \cdot |\mathcal{T}_\ell(\mu)|, \tag{2.1}$$

while if $\lambda = \mu$ *we have:*

$$|\mathcal{O}_\ell(m,[\nu_1])| = \binom{|\mathcal{T}_\ell(\lambda)|+1}{2} \tag{2.2}$$

and

$$|\mathcal{O}_\ell(m,[\nu_1]^*)| = \binom{|\mathcal{T}_\ell(\lambda)|}{2}, \tag{2.3}$$

where $*$ *is defined in (2.4) below.*

Proof. For each $m \ge 1$ and $\ell \ge 6$, we construct an explicit bijection between two kinds of objects: on one side, pairs of restricted tableaux (t_λ, t_μ), where $\lambda, \mu \in \Lambda(m,\ell)$ and $\lambda_1 \ge \mu_1$ and on the other, oscillating tableaux o_ν of length m with shapes restricted to $\Gamma(\ell)$.

On $\Gamma(\ell)$ define a reflection $*$ by:

$$\tilde{\lambda}_1^* = 4 - \tilde{\lambda}_1 \text{ and } \tilde{\lambda}_j^* = \tilde{\lambda}_j \text{ for } j > 1 \tag{2.4}$$

Explicitly, we have $[0]^* = [1^4]$, $[\lambda_1]^* = [\lambda_1, 1^2]$ for $\lambda_1 > 0$ and $[\lambda_1, \lambda_2]^* = [\lambda_1, \lambda_2]$.

For any $(\sigma, \tau) \in \Lambda(m, \ell) \times \Lambda(m, \ell)$ define the following functions:

(1) $f(\sigma, \tau) := \sigma_1 + \tau_1 - m$
(2) $g(\sigma, \tau) := |\sigma_1 - \tau_1|$
(3) $s(\sigma, \tau) := \mathrm{sgn}(\sigma_1 - \tau_1)$

Suppose we are given $\lambda, \mu \in \Lambda(m, \ell)$ with $\lambda_1 \geq \mu_1$, and $(t_\lambda, t_\mu) \in \mathcal{T}_\ell(\lambda) \times \mathcal{T}_\ell(\mu)$, *i.e.*

$$t_\lambda := \lambda^{(0)} = [0] \to \lambda^{(1)} \to \cdots \to \lambda^{(m)} = \lambda$$

and

$$t_\mu := \mu^{(0)} = [0] \to \mu^{(1)} \to \cdots \to \mu^{(m)} = \mu.$$

The basic idea is that the two rows of the jth term in the oscillating tableau associated to λ and μ are obtained by plugging $\lambda^{(j)}$ and $\mu^{(j)}$ into the formulas (1) and (2) above. Sometimes, however, the resulting diagram must be reflected. We now explain the rules for determining when this must be done.

For each j, let m_j denote the maximal positive integer $i \leq j$ such that $s(\lambda^{(i)}, \mu^{(i)}) \neq 0$; if no such integer exists, let $m_j = 0$. We define

$$s^{(j)} = \begin{cases} 1 & \text{if } m_j = 0, \\ s(\lambda^{(m_j)}, \mu^{(m_j)}) & \text{if } m_j \neq 0. \end{cases}$$

We construct an oscillating tableau o_ν as follows. For each j define

$$\nu^{(j)} = \begin{cases} [f(\lambda^{(j)}, \mu^{(j)}), g(\lambda^{(j)}, \mu^{(j)})], & \text{if } s^{(j)} = 1, \\ [f(\lambda^{(j)}, \mu^{(j)}), g(\lambda^{(j)}, \mu^{(j)})]^*, & \text{if } s^{(j)} = -1. \end{cases}$$

Figure 1 illustrates this procedure for a pair (λ, μ) of tableaux of size 8.

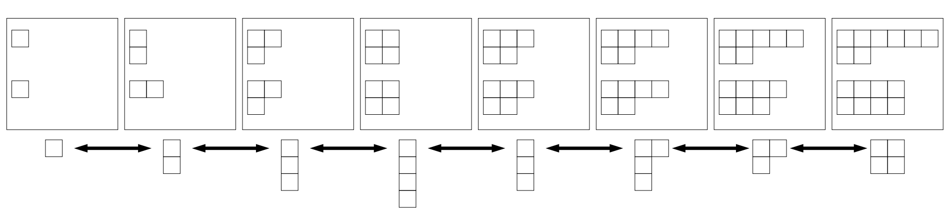

Figure 1

We need to check that

$$[0] = \nu^{(0)} \leftrightarrow \nu^{(1)} \leftrightarrow \cdots \leftrightarrow \nu^{(m)} = \nu$$

with each $\nu^{(j)} \in \Gamma(\ell)$. Since $\Gamma(\ell)$ is closed under $*$, to ensure that $\nu^{(j)} \in \Gamma(\ell)$ it is enough to show that $[f(\lambda^{(j)}, \mu^{(j)}), g(\lambda^{(j)}, \mu^{(j)})] \in \Gamma(\ell)$. This holds because $\tilde{\sigma}_1 + \tilde{\sigma}_2 \le 4$ for every diagram σ with ≤ 2 rows, and

$$\begin{aligned} f(\lambda^{(j)}, \mu^{(j)}) + g(\lambda^{(j)}, \mu^{(j)}) &= 2\max(\lambda_1^{(j)}, \mu_1^{(j)}) - j \\ &= \max(\lambda_1^{(j)} - \lambda_2^{(j)}, \mu_2^{(j)} - \mu_1^{(j)}) \le \ell - 2 \end{aligned}$$

as $\lambda^{(j)}, \mu^{(j)} \in \Lambda(j, \ell)$. To show that $\nu^{(j)} \leftrightarrow \nu^{(j+1)}$ one checks the (four) cases corresponding to the relationships $\lambda^{(j)} \to \lambda^{(j+1)}$ and $\mu^{(j)} \to \mu^{(j+1)}$. This is straightforward, although tedious, with some care needed to see that the relationship $\nu^{(j)} \leftrightarrow \nu^{(j+1)}$ holds if $\nu^{(j)}$ has two rows and $\nu^{(j+1)}$ has three.

Given an oscillating tableau $\nu^{(j)}$ in $\Gamma(\ell)$, we write each $\nu^{(j)}$ as $[\nu_1^{(j)}, \nu_2^{(j)}]$ or $[\nu_1^{(j)}, \nu_2^{(j)}]^*$. Let $s^{(j)}$ be -1 if and only if there is a $*$ and 1 if and only if there is not. This is uniquely defined except when $\nu^{(j)}$ has exactly two rows. In this case, we apply the following rule: if k is the smallest integer greater than or equal to j such that $\nu^{(k)}$ has fewer than 2 or more than 2 rows, then $s^{(j)} = s^{(k)}$; if there is no such k, then $s^{(j)} = 1$. Figure 2 illustrates this procedure:

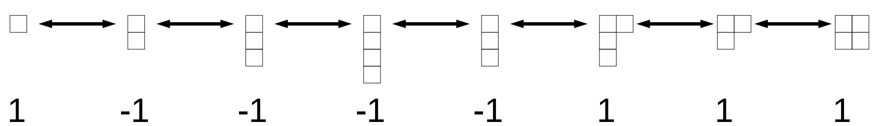

Figure 2

We next define for each j between 1 and m

$$\begin{aligned} \lambda_1^{(j)} &= \frac{j + \nu_1^{(j)} + s^{(j)}\nu_2^{(j)}}{2}, \\ \lambda_2^{(j)} &= j - \lambda_1^{(j)} \\ \mu_1^{(j)} &= \frac{j + \nu_1^{(j)} - s^{(j)}\nu_2^{(j)}}{2}, \\ \mu_2^{(j)} &= j - \mu_1^{(j)}. \end{aligned}$$

These are integers because

$$\nu_1^{(j)} + s^{(j)}\nu_2^{(j)} \equiv \nu_1^{(j)} + \nu_2^{(j)} \equiv |\nu^{(j)}| \equiv j \pmod 2.$$

They are obviously all non-negative, and setting

$$\lambda^{(j)} = [\lambda_1^{(j)}, \lambda_2^{(j)}],\ \mu^{(j)} = [\mu_1^{(j)}, \mu_2^{(j)}],$$

we obtain diagrams in $\Lambda(j,\ell)$. If $\nu^{(m)}$ has exactly two rows, our sign convention guarantees that $s^{(m)} = 1$ and therefore that $\lambda_1 = \lambda_1^{(m)} > \mu_1^{(m)} = \mu_1$; otherwise $\lambda_1 = \mu_1$.

It is not difficult to see that these two constructions are mutually inverse; the most delicate point is that signs $s^{(j)}$ are respected.

The resulting bijection implies equation (2.1) immediately. Equations (2.2) and (2.3) follow by setting $\lambda = \mu$ and defining an order on the tableaux t_λ of shape λ according to which $t_\lambda \geq t_\mu$ if and only if

$$(\lambda_1^{(m)}, \lambda_1^{(m-1)}, \ldots, \lambda_1^{(1)}) \geq (\mu_1^{(m)}, \mu_1^{(m-1)}, \ldots, \mu_1^{(1)})$$

in lexicographic order. Thus $t_\lambda \geq t_\mu$ if and only if $\nu^{(n)}$ has ≤ 1 row and $t_\lambda < t_\mu$ if and only if $\nu^{(n)}$ has ≥ 3 rows.

□

We remark that the above theorem allows us to deduce for each fixed ℓ closed form expressions for $|\mathcal{O}_\ell(m,\lambda)|$ using the corresponding expressions for $|\mathcal{T}_\ell(\lambda)|$. Such formulae can be obtained via the interpretation of $|\mathcal{T}_\ell([m-p,p])|$ as the dimension of a certain representations $V_{m,p}$ and $\overline{V}_{m,p}$ (defined below) of the braid group $\mathcal{B}_m$. For example, we have

Lemma 2.2 (Jones). (a) *For $\ell = \infty$,*

$$|\mathcal{T}([m-p,p])| = \dim V_{m,p} = \binom{m}{p} - \binom{m}{p-1}$$

where $\binom{m}{-1} = 0$ by convention.

(b) *For $\ell = 6$,*

$$|\mathcal{T}_\ell([m-p,p])| = \dim(\overline{V}_{m,p}) = \begin{cases} (3^{\lfloor \frac{m-1}{2} \rfloor} + 1)/2 & m-2p = 0,1 \\ (3^{\lfloor \frac{m-1}{2} \rfloor} - 1)/2 & m-2p = 3,4 \\ 3^{\lfloor \frac{m-1}{2} \rfloor} & m-2p = 2 \end{cases}$$

Proof. The case with $\ell = \infty$ appears explicitly in [J1]. The $\ell = 6$ formulae can be easily proved by induction using the structure of $\Lambda(6)$—see the last example in [J3, §4.2]. □

The following technical lemma shows that pairs (m,λ) with $\lambda \in \Gamma(\ell)$ are distinguished by pairs $(m-1,\nu)$ with $\nu \leftrightarrow \lambda$ as long as $m \geq 3$, and will be used in the proof of Theorem 6.3. Define sets

$$P(m,\lambda) := \{\nu \in \Gamma(\ell) : \nu \leftrightarrow \lambda, \quad (m-1) - |\nu| \in 2\mathbb{N}\}$$

of level $m-1$ predecessors of λ. Then we have:

Lemma 2.3. *Fix $m \geq 3$, and let $\lambda, \mu \in \Gamma(\ell)$ with $m-|\lambda|, m-|\mu| \in 2\mathbb{N}$. Then $\lambda \neq \mu$ implies:*

$$P(m,\lambda) \neq P(m,\mu). \tag{2.5}$$

Proof. First suppose $|\lambda| \geq 3$, $|\mu| \geq 3$, $\nu \to \lambda$ implies $\nu \in \Gamma(\ell)$ and $\nu \to \mu$ implies $\nu \in \Gamma(\ell)$. Then a direct application of [Wz1, Lemma 2.11(b)] shows that (2.5) holds for all λ, μ with these restrictions. The only diagrams that do not satisfy these extra hypotheses are $[0], [1], [2], [1,1]$ and $[\ell-2,1,1]$ (which fails because $[\ell-2,1] \notin \Gamma(\ell)$). The diagram $[\ell-3,1,1]$ is the unique diagram in $\Gamma(\ell)$ such that $[\ell-2,1,1] \leftrightarrow [\ell-3,1,1]$, so clearly (2.5) holds for $\lambda = [\ell-3,1,1]$ and μ arbitrary. The remaining diagrams can be handled similarly, noting that since $m \geq 3$, $[1,1] \leftrightarrow [1,1,1]$, $[2] \leftrightarrow [3]$, $[0] \leftrightarrow [1]$ and $[1]$ is the unique diagram with the latter property. □

3. Temperley-Lieb Algebras

Temperley-Lieb algebras are natural representation spaces for braid groups. They admit a natural trace. The Jones polynomial of a link L is defined as the trace of any braid β for which the corresponding closed braid $\hat{\beta} = L$.

Fix a complex variable q.

Definition 3.1. The Temperley-Lieb algebra $T_m(q)$ is the $\mathbb{C}(q)$-algebra generated by $e_1, e_2, \cdots, e_{m-1}$ satisfying:

(T1) $e_i e_{i\pm1} e_i = \frac{q^{-2}}{(1+q^{-2})^2} e_i = \frac{1}{(q+q^{-1})^2} e_i$
(T2) $e_i e_j = e_j e_i$ for $|i-j| \geq 2$
(H) $e_i^2 = e_i$

By convention we put $T_0 = T_1 = \mathbb{C}(q)$. When there is no danger of confusion we will denote $T_m(q)$ simply by T_m.

Remark 3.2. The reader is warned that our definition of the Temperley-Lieb algebra differs slightly from the standard one (see [GW] for example) in which the Temperley-Lieb algebras are defined with parameter t which corresponds to q^{-2} in our definition.

Lemma 3.3. *Define $g_i := (1+q^{-2})e_i - 1$.*

(a) *The following relations hold in T_m:*
(B1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ *for* $1 \leq i \leq m-2$
(B2) $g_i g_j = g_j g_i$ *if* $|i-j| \geq 2$
(T3) $g_i^{-1} = (q^2+1)e_i - 1$
(T4) $g_i e_i = q^{-2} e_i$
(T5) $e_i g_{i+1} e_i = \frac{-1}{q^{-2}+1} e_i$

(T6) $g_ig_{i\pm 1}g_i + g_ig_{i\pm 1} + g_{i\pm 1}g_i + g_i + g_{i\pm 1} + 1 = 0$
(T7) $(g_i + 1)(g_i - q^{-2}) = 0$.

(b) *The inductive limit of the algebras T_m admits a $\mathbb{C}(q)$-valued trace* tr *uniquely determined by:*
(M1) $\mathrm{tr}(1) = 1$
(M2) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$
(M3) $\mathrm{tr}(ae_{m-1}) = \frac{1}{(q+q^{-1})^2}\mathrm{tr}(a)$ *for* $a \in T_{m-1}$.

The relations (B1) and (B2) imply that T_m is a quotient of $\mathbb{C}(q)\mathcal{B}_m$. Moreover, one deduces from these relations that T_m is finite-dimensional over $\mathbb{C}(q)$.

Specializations of Temperley-Lieb algebras remain well-defined for $q \notin \{0, \pm i\}$ from which we obtain $\mathbb{C}$-algebras and $\mathbb{C}$-representations of $\mathcal{B}_m$ factoring over T_m. The analysis of these specializations breaks naturally into two cases: 1) the generic case–those q for which $q^{2k} - 1 \in \mathbb{C}^*$ for all integers $k \geq 1$ and 2) the proper root of unity case–those q for which q^2 is a primitive ℓth root of unity with $\ell \geq 3$. When we wish to consider both cases simultaneously we say that q^2 is a primitive ℓth root of unity with $3 \leq \ell \leq \infty$ where the case $\ell = \infty$ covers that former case. By an abuse of notation we will continue to denote these specializations by T_m since ℓ and q will always be clear from the context.

In the generic case the the trace tr is faithful, *i.e.* the annihilator ideal $J_m := \{a \in T_m : \mathrm{tr}(ab) = 0 \text{ for all } b \in T_m\} = \{0\}$. Moreover, in these cases the algebras T_m are semisimple.

When q^2 is a primitive ℓth root of unity for $3 \leq \ell < \infty$ the specializations are not semisimple, and tr is not faithful. However, the annihilator of the trace $J_m(q, \ell)$ contains the Jacobson radical and the (semisimple) quotient algebra $T_m/J_m(q, \ell)$ will be denoted by $\overline{T}_m$.

As semisimple finite dimensional algebras, T_m and $\overline{T}_m$ are direct sums of full matrix algebras. The simple subalgebras of T_m and $\overline{T}_m$ are in one-to-one correspondence with the subsets $\Lambda(m, \ell) \subset YD$, where $\ell = \infty$ covers the generic case. The decompositions of T_m and $\overline{T}_m$ into full matrix algebras and the restriction rules are described in the following:

Proposition 3.4. *Define $T_{2,0}$ and $T_{2,1}$ to be the eigenspaces of $g_1 \in T_2$ corresponding to eigenvalues -1 and q^{-2} respectively.*

(a1) *For the generic cases $\ell = \infty$, we have:*

$$T_m = \bigoplus_p T_{m,p}$$

where $0 \leq p \leq \lfloor \frac{m}{2} \rfloor$, and $T_{m,p}$ is a full matrix algebra, corresponding to Young diagram $[m - p, p]$.

(a2) *For any Young diagram $[m-p,p]$, denote by $V_{m,p}$ the T_m-representation such that $T_{m,p} \cong \mathrm{End}(V_{m,p})$. Then the restriction of $V_{m,p}$ to T_{m-1} decomposes irreducibly as*

$$V_{m-1,p} \oplus V_{m-1,p-1}$$

where we set $V_{n,t} = \{0\}$ if $[n-t,t] \notin \Lambda(n,\infty)$.

(b1) *Suppose q^2 is an ℓ root of unity with $3 \leq \ell < \infty$. Then*

$$\overline{T}_m = \bigoplus_p \overline{T}_{m,p}$$

where the sum is over all p such that $[m-p,p] \in \Lambda(m,\ell)$ and $\overline{T}_{m,p}(\ell)$ is a full matrix algebra.

(b2) *Denote by $\overline{V}_{m,p}$ the $\overline{T}_m$-representation such that $\overline{T}_{m,p} \cong \mathrm{End}(\overline{V}_{m,p})$ and set $\overline{V}_{n,t} = \{0\}$ if $[n-t,t] \notin \Lambda(n,\ell)$. Then the restriction of $\overline{V}_{m,p}$ to $\overline{T}_{m-1}$ decomposes irreducibly as:*

$$\overline{V}_{m-1,p} \oplus \overline{V}_{m-1,p-1}$$

where we discard any summand that is $\{0\}$.

The restriction rule given above can be more easily explained combinatorially: the representation $V_{m-1,s}$ appears in the restriction of $V_{m,p}$ to T_{m-1} if and only if $[m-1-s,s] \to [m-p,p]$ with an analogous statement for $\overline{T}_m$. From this description we obtain the *Bratteli diagrams* for T_m and $\overline{T}_m$. These are graphs with vertices labelled by diagrams $[m-p,p] \in \Lambda(\ell)$ (where $\ell = \infty$ covers the generic case as usual) and with an edge between the vertex labelled by $[m-p,p]$ and $[m-1-s,s]$ if and only if $V_{m-1,s}$ is a T_{m-1}-subrepresentation of $V_{m,p}$ with the same statement for $\overline{T}_m$ and $\overline{V}_{m,p}$. The ambiguity with the two components of T_2 is removed by the definition of $T_{2,0}$ and $T_{2,1}$ as above. Observing that $\dim(T_0) = \dim(T_1) = 1$, we can inductively compute the dimensions of $V_{m,p}$ and $\overline{V}_{m,p}$ by counting the increasing paths $t_{[m-p,p]}$ in the Bratteli diagram of T_m or $\overline{T}_m$ from $[0]$ to $[m-p,p]$. Thus the representation spaces $V_{m,p}$ and $\overline{V}_{m,p}$ have bases labelled by such paths:

$$[0] \to [1] \to \cdots \to [m-p,p],$$

where each diagram must be in $\Lambda(\ell)$. But increasing paths in the Bratteli diagrams of T_m and $\overline{T}_m$ are just Young tableaux restricted to $\Lambda(\ell)$ (for $\ell = \infty$ and $6 \leq \ell < \infty$ respectively) so from this we see that

$$\dim V_{m,p} = |\mathcal{T}_\infty([m-p,p])|; \dim \overline{V}_{m,p} = |\mathcal{T}_\ell([m-p,p])|.$$

4. BMW Algebras

While the Jones polynomial $V_L(t)$ was derived from the trace on the Temperley-Lieb algebras, the two-variable Kauffmann polynomial $F_L(a,z)$ [K] was first defined in a purely combinatorial way. However, not long after its definition, Birman-Wenzl and Murakami ([BWz], [M]) independently found the appropriate traced quotients of the braid group algebras corresponding to $F_L(a,z)$, and they are now known as BMW (or q-Brauer) algebras. The reader is warned that the parameters r and q below correspond to a different version $K_L(r,q)$ of the Kauffmann polynomial related to $F_L(a,z)$ by a non-trivial change of variables.

4.1. **Definitions and Algebraic Results.**

Definition 4.1. The BMW algebra $\mathcal{C}_m(r,q)$ is the $\mathbb{C}(r,q)$-algebra with invertible generators $G_1, G_2, \ldots, G_{m-1}$ satisfying the braid relations $(B1)$ and $(B2)$ above and:

(R1) $(G_i - r^{-1})(G_i - q)(G_i + q^{-1}) = 0$
(R2) $E_i G_{i-1}^{\pm 1} E_i = r^{\pm 1} E_i$, where
(E) $(q - q^{-1})(1 - E_i) = G_i - G_i^{-1}$ defines E_i.

By convention $\mathcal{C}_0(r,q) = \mathcal{C}_1(r,q) = \mathbb{C}(r,q)$. These relations imply:

Proposition 4.2. [BWz, §3]

(a) *The algebra* $\mathcal{C}_m(r,q)$ *is linearly spanned by elements of the form* $a\chi b$ *where* $a, b \in \mathcal{C}_{m-1}(r,q)$ *are monomials and* $\chi \in \{1, G_{m-1}, E_{m-1}\}$.
(b) *The elements in* $\mathcal{C}_m(r,q)$ *spanned by monomials of the form* $aE_{m-1}b$ *with* $a, b \in \mathcal{C}_{m-1}(r,q)$ *form an ideal* $\mathcal{I}_m$, *and* $\mathcal{C}_m(r,q)/\mathcal{I}_m$ *is isomorphic to the Hecke algebra* $\mathcal{H}_m(q^2)$.

The inductive limit of the algebras $\mathcal{C}_m(r,q)$ is equipped with a trace:

Proposition 4.3. [Wz2, Lemma 3.4] *Set* $x = \frac{r - r^{-1}}{q - q^{-1}} + 1$. *There exists a functional,* Tr, *on* $\mathcal{C}_\infty(r,q)$ *uniquely defined inductively by:*

(1) $\mathrm{Tr}(1) = 1$
(2) $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$
(3) $\mathrm{Tr}(E_i) = 1/x$
(4) $\mathrm{Tr}(G_i^{\pm 1}) = r^{\pm 1}/x$
(5) $\mathrm{Tr}(a\chi b) = \mathrm{Tr}(\chi)\mathrm{Tr}(ab)$ *for* $a, b \in \mathcal{C}_{m-1}(r,q)$, $\chi \in \{G_{m-1}, E_{m-1}\}$.

As in the case of Temperley-Lieb algebras, one may specialize r and q to be complex numbers and for any specialization for which $\mathcal{C}_m(r,q)$ and Tr are well-defined both Propositions 4.2 and 4.3 still hold. For such r and q denote the annihilator ideal of Tr on $\mathcal{C}_m(r,q)$ by: $A_m(r,q)$.

As long as $r \neq \pm q^n$ for any integer n and $q^{2k}-1 \in \mathbb{C}^*$ for all $k \geq 1$, the trace Tr is faithful on $\mathcal{C}_m(r,q)$; moreover, $\mathcal{C}_m(r,q)$ is semisimple. We will shorten $\mathcal{C}_m(r,q)$ to $\mathcal{C}_m$ for these generic cases.

The specializations of BMW algebras with $r = \pm q^n$ are related (via quantum Schur-Weyl-Brauer duality) to quantum groups of Lie types B, C and D, while if we further specialize q^2 to be a primitive ℓth root of unity we obtain interesting $\mathbb{C}$-representations of $\mathcal{B}_m$ in analogy with the Temperley-Lieb situation. When $r = q^n$ and/or q^2 is a root of unity, the BMW algebras fail to be semisimple. By taking the quotient by the ideal $A_m(r,q)$ semisimplicity can often be recovered. For example,

Proposition 4.4 ([Wz2])**.** *Fix r and q with $r = q^n$ where $3 \leq n \leq \ell-3$ and q^2 is a primitive ℓth root of unity with $\ell \leq \infty$. Then $\overline{\mathcal{C}}_m(r,q) := \mathcal{C}_m(r,q)/A_m(r,q)$ is semisimple.*

As usual, we designate the case where $q^{2k}-1 \in \mathbb{C}^*$ for all $k \geq 1$ by $\ell = \infty$.

4.2. **Representation Theory.** In this paper we are interested in the cases where $r = q^3$ and q^2 is a primitive ℓth root of unity with $\ell \leq \infty$. As described in [LRW] Prop. 6.2 (1)(c), the $\mathcal{B}_m$-representations factoring over $\mathcal{C}_m(q^3,q)$ are non-degenerate provided q^2 is an ℓth root of unity with $6 \leq \ell \leq \infty$. The simple subalgebras of $\mathcal{C}_m$ are in one-to-one correspondence with Young diagrams λ with $m-|\lambda| \in 2\mathbb{N}$, while for the semisimple quotients of the specializations of $\overline{\mathcal{C}}_m(r,q)$ with $r = q^3$ and q^2 an ℓth root of unity with $6 \leq \ell \leq \infty$ one must restrict to diagrams in the set $\Gamma(\ell)$ defined above.

We have the following description of the simple decompositions and restriction rules for BMW algebras in both the generic case and the specializations we study:

Proposition 4.5. *Define $\mathcal{C}_{2,[0]}$, $\mathcal{C}_{2,[1,1]}$ and $\mathcal{C}_{2,[2]}$ to be the eigenspaces of $G_1 \in \mathcal{C}_2$ corresponding to eigenvalues r^{-1}, $-q^{-1}$ and q respectively.*

(a1) *Suppose $r \neq \pm q^n$ and q^2 is not a root of unity. Then $\mathcal{C}_m = \bigoplus_\lambda \mathcal{C}_{m,\lambda}$ where $m-|\lambda| \in 2\mathbb{N}$, and $\mathcal{C}_{m,\lambda}$ is a full matrix algebra.*

(a2) *If $W_{m,\lambda}$ is a simple $\mathcal{C}_{m,\lambda}$-module then the restriction of $W_{m,\lambda}$ to $\mathcal{C}_{m-1}(r,q)$ decomposes irreducibly as:*

$$\bigoplus_{\mu \leftrightarrow \lambda} W_{m-1,\mu}.$$

(b1) *Suppose $r = q^3$ and q^2 is an ℓth root of unity with $6 \leq \ell \leq \infty$. Then*

$$\overline{\mathcal{C}}_m(r,q) = \bigoplus_{\lambda} \overline{\mathcal{C}}_{m,\lambda}(r,q)$$

where $m-|\lambda| \in 2\mathbb{N}$ and $\lambda \in \Gamma(\ell)$ and $\overline{\mathcal{C}}_{m,\lambda}(r,q)$ is a full matrix algebra.

(b2) *Let $\overline{W}_{m,\mu}$ be a simple $\overline{\mathcal{C}}_m(r,q)$ module with r and q as in (b1). Then the restriction of $\overline{W}_{m,\mu}$ to $\overline{\mathcal{C}}_{m-1}(r,q)$ decomposes irreducibly as:*

$$\bigoplus_{\substack{\mu \leftrightarrow \lambda \\ \mu \in \Gamma(\ell)}} \overline{W}_{m-1,\mu}.$$

This description gives us a convenient way of encoding the inclusions of BMW algebras via their Bratteli diagrams. The ambiguity between the three simple components for $m = 2$ is removed by assigning the labels to eigenspaces as in the proposition above. Define a graph whose vertices are labelled by (m, λ) where $m-|\lambda| \in 2\mathbb{N}$ and the labels (m, λ) and $(m-1, \mu)$ are connected by an edge if and only if $\lambda \leftrightarrow \mu$. For specializations of $\overline{\mathcal{C}}_m(r,q)$ with $r = q^3$ and q^2 and ℓth root of unity with $6 \leq \ell \leq \infty$, the Bratteli diagram is defined in the same way except that the Young diagrams are restricted to be in the set $\Gamma(\ell)$ defined in Section 2. From this we see that there are bases for $W_{m,\lambda}$ and $\overline{W}_{m,\lambda}$ indexed by the set of paths of length m in the Bratteli diagram beginning at $[0]$ and ending at λ (where all diagrams must be in $\Gamma(\ell)$ in the latter case). From the structure of the Bratteli diagrams we see that these paths are in one-to-one correspondence with oscillating tableaux. Thus, the dimension of $W_{m,\lambda}$ (respectively, $\overline{W}_{m,\lambda}$) is the number $|\mathcal{O}(m,\lambda)|$ of oscillating tableaux (resp. $|\mathcal{O}_\ell(m,\lambda)|$) of shape λ and length m. Note that when $r = q^3$ and q^2 is a primitive ℓth root of unity the Bratteli diagram for $\overline{\mathcal{C}}_m(r,q) \subset \overline{\mathcal{C}}_{m+1}(r,q)$ depends only on ℓ, not on the specific choice of q.

5. Symmetric Squares of Algebras

Let A be an associative $\mathbb{C}$-algebra. We define $\mathcal{S}^2 A$ to be the subalgebra of $A \otimes_{\mathbb{C}} A$ generated by $\{a \otimes a \mid a \in A\}$.

Lemma 5.1. *Let $A = A_1 \oplus \cdots \oplus A_n$. Then*

$$\mathcal{S}^2 A = \bigoplus_{i=1}^{n} \mathcal{S}^2 A_i \oplus \bigoplus_{j=1}^{n-1} \bigoplus_{k=j+1}^{n} A_j \otimes A_k.$$

Proof. Let $\sigma_{jk} \colon A_j \otimes A_k \to A_k \otimes A_j$ exchange factors. The natural inclusion

$$\iota \colon (x_i, y_{jk}) \mapsto \sum_{i=1}^{n} x_i + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} (y_{jk} + \sigma_{jk}(y_{jk}))$$

is obviously injective. It is surjective because for all $a = a_1 + \cdots + a_n$, with $a_i \in A_i$, we have

$$a \otimes a = \iota(a_1 \otimes a_1, \ldots, a_n \otimes a_n, a_1 \otimes a_2, \ldots, a_{n-1} \otimes a_n).$$

□

Proposition 5.2. *If $A = M_n(\mathbb{C})$, then*

$$\mathcal{S}^2 A = M_{\binom{n+1}{2}}(\mathbb{C}) \oplus M_{\binom{n}{2}}(\mathbb{C}).$$

If $B = M_m(\mathbb{C})$, then $A \otimes B = M_{mn(\mathbb{C})}$.

Proof. The proposition is trivial when $n = 1$ (where $M_0(\mathbb{C})$ is understood to mean the zero-ring). We therefore assume $n \geq 2$.

If $V = \mathbb{C}^n$, then $\mathrm{GL}_n(\mathbb{C})$ acts on V, and $V \otimes V$ decomposes as a direct sum of two irreducible $\mathrm{GL}_n(\mathbb{C})$-representations: $\mathcal{S}^2 V$ and $\wedge^2 V$. Thus, the diagonal image of $\mathrm{GL}_n(\mathbb{C})$ in $\mathrm{End}(V \otimes V)$ lies in

$$\mathrm{End}(\mathcal{S}^2 V) \oplus \mathrm{End}(\wedge^2 V) \cong M_{\binom{n+1}{2}}(\mathbb{C}) \oplus M_{\binom{n}{2}}(\mathbb{C}).$$

As $\mathrm{GL}_n(\mathbb{C})$ is dense in $M_n(\mathbb{C})$, the same is true of the diagonal image of $M_n(\mathbb{C})$, and it follows that the subalgebra of $\mathrm{End}(V \otimes V)$ generated by $a \otimes a$ for $a \in M_n(\mathbb{C})$ is contained in $\mathrm{End}(\mathcal{S}^2 V) \oplus \mathrm{End}(\wedge^2 V)$. Conversely, if $\mathcal{S}^2 M_n(\mathbb{C})$ is a $*$-subalgebra of $M_{n^2}(\mathbb{C})$ and therefore a semisimple algebra. If it is properly contained in $\mathrm{End}(\mathcal{S}^2 V) \oplus \mathrm{End}(\wedge^2 V)$, then it has a larger centralizer in $\mathrm{End}(V \otimes V)$, so the centralizer of the diagonal image of $\mathrm{GL}_n(\mathbb{C})$ in $\mathrm{End}(V \otimes V)$ has dimension > 2. This is impossible by Schur's lemma; we have already observed that $V \otimes V$ decomposes as the sum of two inequivalent irreducible representations of $\mathrm{GL}_n(\mathbb{C})$.

For the second claim, let $W = \mathbb{C}^m$. There is a natural map $\mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W)$ which is an isomorphism since

$$\begin{aligned} \mathrm{End}(V) \otimes \mathrm{End}(W) &= (V \otimes V^*) \otimes (W \otimes W^*) \\ &= V \otimes W \otimes V^* \otimes W^* = \mathrm{End}(V \otimes W). \end{aligned}$$

□

Remark 5.3. A natural setting in which to consider symmetric squares of algebras is that of von Neumann algebras. The second part of Proposition 5.2 is well known to hold for factors. We do not know whether the first part holds as well, i.e., whether the symmetric square of a non-trivial factor is always the direct sum of two factors.

If A is endowed with a linear functional $\mathrm{tr}\colon A \to \mathbb{C}$ satisfying the trace identity $\mathrm{tr}(ab) = \mathrm{tr}(ba)$, then $\mathrm{tr} \otimes \mathrm{tr}\colon A \otimes A \to \mathbb{C}$ also satisfies the trace identity, so the same is true of its restriction (denoted tr^2) to $\mathcal{S}^2 A$.

We apply the symmetric square construction to Temperley-Lieb algebras. As $T_m = \bigoplus_i T_{m,i}$ is a direct sum of full matrix algebras, the symmetric square $\mathcal{S}^2 T_m$ can be decomposed as:

$$\bigoplus_i \mathcal{S}^2 T_{m,i} \oplus \bigoplus_{j<k} T_{m,j} \otimes T_{m,k}.$$

Thus $V_{m,j} \otimes V_{m,k}$ $(j < k)$, $\mathcal{S}^2 V_{m,i}$ and $\bigwedge^2 V_{m,j}$ are irreducible representations of $\mathcal{S}^2 T_m$. The trace tr on T_m determines the trace tr^2 on $\mathcal{S}^2 T_m$.

Observe that the same analysis applies to the symmetric square $\mathcal{S}^2 \overline{T}_m$ for $\ell < \infty$ with analogous conclusions replacing $V_{m,i}$ by $\overline{V}_{m,i}$ and restricting to $\Lambda(\ell)$-diagrams in all formulae.

Fix q such that q^2 is an ℓth root of unity with $6 \leq \ell \leq \infty$ and set $r = q^3$ and $x = \frac{r-r^{-1}}{q-q^{-1}} + 1 = (q+q^{-1})^2$. By an abuse of notation we will continue to denote the images of the generators of T_m in $\overline{T}_m$ by g_i and e_i as this should cause no confusion. We define elements $\tilde{G}_i = q(g_i \otimes g_i)$ and $\tilde{E}_i := x(e_i \otimes e_i)$ of $\mathcal{S}^2 T_m$ or $\mathcal{S}^2 \overline{T}_m$ and derive some relations from those of T_m:

Lemma 5.4. *We have the following identities:*

(1) $(\tilde{G} - r^{-1})(\tilde{G} - q)(\tilde{G} + q^{-1}) = 0$
(2) $(q - q^{-1})(1 - \tilde{E}) = (\tilde{G} - \tilde{G}^{-1})$
(3) $\tilde{E}_i \tilde{G}_{i-1}^{\pm 1} \tilde{E}_i = r^{\pm 1} \tilde{E}_i$
(4) $\mathrm{tr}^2(1) = 1$
(5) $\mathrm{tr}^2(ab) = \mathrm{tr}^2(ba)$
(6) $\mathrm{tr}^2(\tilde{E}_i) = 1/x$
(7) $\mathrm{tr}^2(\tilde{G}_i) = r^{\pm 1}/x$
(8) $\mathrm{tr}^2(a\chi b) = \mathrm{tr}^2(\chi) tr^2(ab)$ *for* $a, b \in \mathcal{S}^2 T_m$, $\chi \in \{\tilde{G}_m, \tilde{E}_m\}$

Proof. All of these relations follow directly from Lemma 3.3. For example, let us prove (2).

$$\begin{aligned}
\tilde{G}_i - \tilde{G}_i^{-1} &= q(g_i \otimes g_i) - (g_i^{-1} \otimes g_i^{-1})/q \\
&= q((q^{-2}+1)^2 e_i \otimes e_i - (q^{-2}+1)(1 \otimes e_i + e_i \otimes 1) + 1 \otimes 1) \\
&\quad - q^{-1}((q^2+1)^2 e_i \otimes e_i - (q^2+1)(1 \otimes e_i + e_i \otimes 1) + 1 \otimes 1) \\
&= (q(q^{-2}+1)^2 - (q^2+1)^2/q) e_i \otimes e_i + (q - q^{-1}) 1 \otimes 1 \\
&= (q - q^{-1})(1 \otimes 1 - (q+q^{-1})^2 e_i \otimes e_i) \\
&= (q - q^{-1})(1 - \tilde{E}_i)
\end{aligned}$$

□

6. The Isomorphism

Throughout this section set $r = q^3$ and let q^2 be an ℓth root of unity with $6 \leq \ell \leq \infty$. Consider the mapping:

$$\Phi(G_i) = \tilde{G}_i, \quad 1 \leq i \leq m-1$$

It is immediate from Lemma 5.4 and the defining relations of $\mathcal{C}_m(r,q)$ that Φ extends to an algebra homomorphism $\mathcal{C}_m(r,q) \to \mathcal{S}^2 T_m$. Another consequence of Lemma 5.4 is that $\Phi(E_i) = \tilde{E}_i$. We can now prove:

Lemma 6.1. *The induced map*

$$\overline{\Phi} : \overline{\mathcal{C}}_m(r,q) = \mathcal{C}_m(r,q)/A_m(r,q) \to \Phi(\mathcal{C}_m(r,q))/\Phi(A_m(r,q))$$

is injective.

Proof. It is enough to show that $\ker \Phi \subset A_m(r,q)$. First note that tr^2 induces a trace form $\Phi^{-1}(\mathrm{tr}^2)$ on $\mathcal{C}_m(r,q)$ that has the Markov property and the values of $\Phi^{-1}(\mathrm{tr}^2)$ and Tr coincide on $\{1, E_i, G_i\}$ for all i so that the uniqueness of Tr implies that $\Phi^{-1}(\mathrm{tr}^2) = \mathrm{Tr}$. Suppose $a \in \ker \Phi$, and $b \in \mathcal{C}_m(r,q)$. Then $\mathrm{Tr}(ab) = \mathrm{tr}^2(\Phi(ab)) = \mathrm{tr}^2(0) = 0$ so that $a \in A_m(r,q)$. □

An immediate corollary of this lemma is that $\overline{\mathcal{C}}_m(r,q)$ is isomorphic to a semisimple quotient of the subalgebra of $\mathcal{S}^2 T_m$ or $\mathcal{S}^2 \overline{T}_m$ generated by $\{\tilde{G}_i\}$ so that $\dim(\overline{\mathcal{C}}_m(r,q)) \leq \dim(\mathcal{S}^2 T_m)$ for $\ell = \infty$ and $\dim(\overline{\mathcal{C}}_m(r,q)) \leq \dim(\mathcal{S}^2 \overline{T}_m)$ for $6 \leq \ell < \infty$. But by Theorem 2.1,

$$\begin{aligned}
\dim(\overline{\mathcal{C}}_m(r,q)) &= \sum_{\substack{\nu \in \Gamma(\ell) \\ m-|\nu| \in 2\mathbb{N}}} |\mathcal{O}_\ell(m,\nu)|^2 \\
&\geq \sum_{p<q\leq \frac{m}{2}} |\mathcal{T}_\ell([m-p,p])|^2 \cdot |\mathcal{T}_\ell([m-r,r])|^2 \\
&+ \sum_{p \leq \frac{m}{2}} \binom{1 + |\mathcal{T}_\ell([m-p,p])|}{2}^2 \\
&+ \sum_{p \leq \frac{m}{2}} \binom{|\mathcal{T}_\ell([m-p,p])|}{2}^2 \\
&= \begin{cases} \dim(\mathcal{S}^2 T_m) & \ell = \infty \\ \dim(\mathcal{S}^2 \overline{T}_m) & 6 \leq \ell < \infty, \end{cases}
\end{aligned}$$

so by dimension we have our main result:

Theorem 6.2. *Let $r = q^3$. Then $\overline{\mathcal{C}}_m(r,q) \cong \mathcal{S}^2T_m$ if q is not a root of unity and $\overline{\mathcal{C}}_m(r,q) \cong \mathcal{S}^2\overline{T}_m$ for q^2 an ℓth root of unity with $6 \leq \ell < \infty$.*

Although we have established isomorphisms between these semisimple algebras as promised, we have not identified the images of the simple components of $\overline{\mathcal{C}}_m(r,q)$ under $\overline{\Phi}$. Not surprisingly, the combinatorial correspondence in Theorem 2.1 is compatible with $\overline{\Phi}$:

Theorem 6.3. *Fix $[m-s,s],[m-t,t] \in \Lambda(m,\ell)$ with and define ν_1 and ν_2 as in Theorem 2.1. Then the map $\overline{\Phi}$ induces isomorphisms of simple algebras as follows for $6 \leq \ell < \infty$:*

(1) *if $\lambda \neq \mu$, $\overline{\mathcal{C}}_{m,[\nu_1,\nu_2]}(r,q) \cong \mathrm{End}(\overline{V}_{m,s}) \otimes \mathrm{End}(\overline{V}_{m,t})$*
(2) *if $\lambda = \mu$, $\overline{\mathcal{C}}_{m,[\nu_1]}(r,q) \cong \mathrm{End}(\mathcal{S}^2\overline{V}_{m,s})$,*
(3) *if $\lambda = \mu$, $\overline{\mathcal{C}}_{m,[\nu_1]^*}(r,q) \cong \mathrm{End}(\wedge^2\overline{V}_{m,s})$*

The same statement holds for $\ell = \infty$ replacing $\overline{\mathcal{C}}_{m,\lambda}$ and $\overline{V}$ by $\mathcal{C}_{m,\lambda}$ and V respectively.

Proof. The cases $m = 0, 1$ are clear since all algebras in question are isomorphic to $\mathbb{C}$. For $m \geq 2$ we proceed by induction on m. The (base) case $m = 2$ follows by checking that the labelling conventions for the eigenspaces of $\tilde{G}_1 \in \mathcal{S}^2T_2$ (induced from those of $g_1 \in T_2$) and $G_1 \in \mathcal{C}_2$ are compatible with the correspondence of Theorem 2.1. Now suppose that the statement holds for some $m-1 \geq 2$. By Theorem 4.5 and Lemma 2.3 any simple component of $\overline{\mathcal{C}}_m(r,q)$ is determined by the set of labels of the simple $\overline{\mathcal{C}}_{m-1}(r,q)$-subalgebras contained in $\overline{\mathcal{C}}_m(r,q)$. Applying the induction hypothesis to these simple $\overline{\mathcal{C}}_{m-1}(r,q)$-subalgebras we obtain isomorphisms between the simple components of $\mathcal{S}^2\overline{T}_{m-1}$ and those of $\overline{\mathcal{C}}_{m-1}(r,q)$ as in the statement of the theorem. Tracing through the corresponding labels of the simple components we see that this implies the result for m. □

7. Braid Group Images

The irreducible representations of $\mathcal{B}_m$ factoring over $\overline{T}_m$ are unitary if $q = e^{\pm\pi i/\ell}$. The closed images of these unitary $\mathcal{B}_m$-representations have been classified in [J1], [BWj] and [FLW]. Our goal in this section is to solve analogous problem for BMW algebras when $r = q^3$. This was the original motivation of this paper. The question of unitarity for representations of $\mathcal{B}_m$ factoring over $\mathcal{C}_m(r,q)$ is not so simple in general. It was shown in [R1] that the cases $r = q^n$ with $n < 0$ even and q *any* primitive ℓth root of unity with ℓ odd can fail to yield unitary representations of $\mathcal{B}_m$. However, Wenzl [Wz2] showed that for essentially all other $r = q^n$ with $q = e^{\pm\pi i/\ell}$ one obtains unitary $\mathcal{B}_m$

representations; in particular, this is so for $r = q^3$ and $q = e^{\pm\pi i/\ell}$ with $6 \leq \ell$.

Throughout this section, we will fix an integer $\ell \geq 6$ and assume $q = e^{\pm\pi i/\ell}$ and $r = q^3$. By Proposition 4.5, we have a decomposition

$$\mathcal{C}_m(r,q) = \bigoplus_{\lambda\in\Gamma(\ell)} \mathrm{End}(\overline{W}_{m,\lambda}).$$

Let $\sigma_{m,\lambda}\colon \mathcal{B}_m \to \bar{\mathrm{GL}}(\overline{W}_{m,\lambda})$ denote the corresponding representation and $\bar{\sigma}_{m,\lambda}\colon \mathcal{B}_m \to \mathrm{P}\bar{\mathrm{GL}}(\overline{W}_{m,\lambda})$ its projectivization. Because of our choice of (q,r), $\sigma_{m,\lambda}$ is always unitary. The topological closure $\bar{H}_{m,\lambda}$ of $\bar{\sigma}_{m,\lambda}(\mathcal{B}_m)$ is therefore a compact Lie group.

We recall that by Proposition 3.4,

$$\overline{T}_m = \bigoplus_s \mathrm{End}(\overline{V}_{m,s}),$$

where $(\rho_{m,s}, \overline{V}_{m,s})$ is the Jones representation corresponding to the Young diagram $[m-s,s]$. Let $\bar{G}_{m,s}$ denote the closure of the image of the projectivized Jones representation $\bar{\rho}_{m,s}(\mathcal{B}_m)$. As $\rho_{m,s}$ is unitary, this is a compact Lie group.

We recall the precise result:

Proposition 7.1. *Let $m \geq 3$ and $\ell \geq 6$ be integers, and let $q = e^{\pm\pi i/\ell}$. Let $1 \leq s \leq m/2$, and set $d_{m,s} := \dim \overline{V}_{m,s}$. Then*

(a1) [BWj, FLW] *If $\ell = 6$ and m is odd, then $d_{m,s} = \frac{3^{(m-1)/2}\pm 1}{2}$ and $\bar{G}_{m,s} \cong \mathrm{PSp}_{m-1}(3)$.*

(a2) [BWj] *If $\ell = 6$ and m is even, then $d_{m,s} \in \{\frac{3^{(m-2)/2}\pm 1}{2}, 3^{\frac{m-2}{2}}\}$ and*

$$\bar{G}_{m,s} \cong \begin{cases} \mathrm{PSp}_{m-2}(3) \ltimes (\mathbb{Z}_3)^{m-2} & s = m/2-1 \\ \mathrm{PSp}_{m-2}(\mathbb{Z}_3) & s \in \{m/2-2, m/2\}. \end{cases}$$

(b) [J1] *If $\ell = 10$ then $\bar{G}_{3,1} \cong \bar{G}_{4,2} \cong A_5$.*

(c) [FLW] *Except in cases (a) and (b), $\bar{G}_{m,s} = \mathrm{PSU}(d_{m,s})$.*

(d) [FLW] *If $s \neq t$, then the $\mathcal{B}_m$-representation $\rho_{m,t}$ is not equivalent to $\chi \otimes \rho_{m,s}$ or $\chi \otimes \rho^*_{m,s}$ for any character χ.*

Remark 7.2. Parts (a) and (b) of the proposition hold for all primitive roots of unity q, not just for the specific values in the statement since the groups in question are finite.

Thanks to Theorem 6.3, up to tensor product with a 1-dimensional representation of $\mathcal{B}_m$, we can identify each $\overline{W}_{m,\lambda}$ with a representation of the form $\mathcal{S}^2\overline{V}_{m,s}$, $\wedge^2\overline{V}_{m,s}$, or $\overline{V}_{m,s} \otimes \overline{V}_{m,t}$. Note that tensoring with a 1-dimensional representation does not affect $\bar{H}_{m,\lambda}$. The image of a

group in $\mathrm{PGL}(V)$ (resp. when $\dim V \geq 3$) is the same as its image in $\mathrm{PGL}(\mathcal{S}^2 V)$ (resp. $\mathrm{PGL}(\wedge^2 V)$), since the natural homomorphisms $\mathrm{PGL}(d) \to \mathrm{PGL}\big(\binom{d+1}{2}\big)$ and $\mathrm{PGL}(d) \to \mathrm{PGL}\big(\binom{d}{2}\big)$ are injective for $d \geq 1$ (resp. $d \geq 3$). To identify the closed images of $\mathcal{B}_m$ under the projectivized tensor products of Jones representations, we combine Proposition 7.1 with Goursat's lemma:

Lemma 7.3 ([Gt])**.** *Suppose $H \subset G_1 \times G_2$ such that the compositions $H \hookrightarrow G_1 \times G_2 \to G_1$ and $H \hookrightarrow G_1 \times G_2 \to G_2$ are surjective homomorphisms. There there exist normal subgroups $N_i \lhd G_i$ and an isomorphism $\psi : G_1/N_1 \to G_2/N_2$ such that H is the graph of ψ,* i.e. *$(g_1, g_2) \in H$ if and only if $\psi(g_1 N_1) = g_2 N_2$.*

Suppose V_1 and V_2 are representation spaces of irreducible unitary representations of $\mathcal{B}_m$. Tensor product defines a natural injective map $\mathrm{PGL}(V_1) \times \mathrm{PGL}(V_2) \to \mathrm{PGL}(V_1 \otimes V_2)$. If H, G_1 and G_2 denote the closure of the image of $\mathcal{B}_m$ in $\mathrm{PGL}(V_1 \otimes V_2)$, $\mathrm{PGL}(V_1)$ and $\mathrm{PGL}(V_2)$ respectively, then $H \hookrightarrow G_1 \times G_2$ satisfies the hypotheses of the lemma.

Theorem 7.4. *Let $\ell \geq 6$, $m \geq 1$, and $\lambda \in \Gamma(\ell)$ be such that $m - |\lambda|$ is a non-negative even integer. Let*

$$\Sigma_6 = \{[4], [4,1,1], [1,1,1,1], [2,2], [1,1], [0]\}.$$

Then,

(7.1)

$$\bar{H}_{m,\lambda} \cong \begin{cases} \{1\} & \lambda = [m], \\ \{1\} & m = 2, \\ \{1\} & m = 3 \text{ and } \lambda = [1,1,1], \\ \{1\} & m = 4 \text{ and } \lambda = [1,1,1,1], \\ \mathrm{PSp}_{m-1}(3) & \ell = 6 \text{ and } m \text{ is odd}, \\ \mathrm{PSp}_{m-2}(3) & \ell = 6 \text{ and } \lambda \notin \Sigma_6, \\ \mathrm{PSp}_{m-2}(3) \ltimes (\mathbb{Z}_3)^{m-2} & \ell = 6, \\ A_5 & \ell = 10,\ m = 3, \text{ and } \lambda \in \{[2,1],[1]\}, \\ A_5 & \ell = 10,\ m = 4, \text{ and } \lambda \in \{[2,2],[0]\}, \\ A_5 \times \mathrm{PSU}(3) & \ell = 10,\ m = 4, \text{ and } \lambda = [1,1], \\ \mathrm{PSU}(d_{m,\frac{m-\lambda_1}{2}}) & \lambda = [\lambda_1], \\ \mathrm{PSU}(d_{m,\frac{m-\lambda_1}{2}}) & \lambda = [\lambda_1,1,1], \\ \mathrm{PSU}(d_{m,\frac{m}{2}}) & \lambda = [1,1,1,1], \end{cases}$$

Here we employ the convention that each condition is assumed to exclude all previous ones, so that for example the sixth case implicitly

requires that m is even. In the generic case, when none of these conditions applies, we have

$$\bar{H}_{m,\lambda} = \mathrm{PSU}(d_{m,\frac{m-\lambda_1+\lambda_2}{2}}) \times \mathrm{PSU}(d_{m,\frac{m-\lambda_1-\lambda_2}{2}}). \tag{7.2}$$

Proof. To begin with, we note that the first four cases of (7.1) are precisely those for which $\dim \overline{W}_{m,\lambda} = 1$, so we may now assume $\dim \overline{W}_{m,\lambda} > 1$. If $\overline{W}_{m,\lambda}$ is the symmetric square of some $\overline{V}_{m,p}$, then $\bar{H}_{m,\lambda} \cong \bar{G}_{m,s}$. Since $\binom{d}{2} > 1$, the exterior square map is injective, so the previous remark applies also in this case. These two remarks account for the last three cases of (7.1) as well as various subcases of the fifth, sixth, seventh, eighth, and ninth cases.

The remaining difficulty is to determine $\bar{H}_{m,\lambda}$ in the tensor product case, when we know it is a subgroup of $\bar{G}_{m,s} \times \bar{G}_{m,t}$ mapping onto each factor. When the two factors are simple and non-isomorphic, then $\bar{H}_{m,\lambda}$ must be the whole product. This is the situation in the tenth case of (7.1) and in (7.2), when

$$d_{m,\frac{m-\lambda_1+\lambda_2}{2}} \neq d_{m,\frac{m+\lambda_1+\lambda_2}{2}}.$$

If both factors are simple and isomorphic and $\bar{H}_{m,\lambda}$ is not the whole product, then it must be the graph of an isomorphism. Every automorphism of $\mathrm{PSU}(d)$ is either inner or the product of an inner automorphism with transpose inverse. Therefore, if $\bar{G}_{m,s} \cong \bar{G}_{m,t} \cong \mathrm{PSU}(d)$ and $\bar{H}_{m,\lambda}$ is the graph of an isomorphism, the representation $\rho_{m,s}$ must be equivalent (up to tensoring by a 1-dimensional representation) to $\rho_{m,t}$ or its dual. This is impossible by Proposition 7.1, so this finishes the case that $\bar{G}_{m,s}$ and $\bar{G}_{m,t}$ are both infinite.

The only remaining cases are those where $\bar{G}_{m,s}$ and $\bar{G}_{m,t}$ are both finite and non-trivial. This cannot happen if $\ell = 10$ (since for each m there is at most one non-trivial value of s which give non-trivial finite image). It can happen only if $\ell = 6$. Here we know [BWj] that the whole image of $\mathcal{B}_m$ in $\overline{T}_m$ is a central extension of either $\mathrm{PSp}_{m-1}(3)$ or $\mathrm{PSp}_{m-2}(3) \ltimes (\mathbb{Z}_3)^{m-2}$ depending on whether m is odd or even. The tensor product of any $\overline{V}_{m,s}$ and $\overline{V}_{m,t}$ is contained in the symmetric square of $\overline{T}_m$, so the image of $\mathcal{B}_m$ in the projectivization of any such tensor product is a quotient of $\mathrm{PSp}_{m-1}(3)$ or $\mathrm{PSp}_{m-2}(3) \ltimes (\mathbb{Z}_3)^{m-2}$ respectively. When m is odd, we therefore automatically have the fifth case of (7.1). By Proposition 7.1, when m is even, $s = m/2-1$ gives $\bar{G}_{m,s} = \mathrm{PSp}_{m-2}(3) \ltimes (\mathbb{Z}_3)^{m-2}$ and the other two values, $s = m/2$ and $s = m/2-2$, give $\bar{G}_{m,s} = \mathrm{PSp}_{m-2}(3)$. This means that if $s = m/2-2$, $t = m/2$, $\bar{H}_{m,\lambda} = \mathrm{PSp}_{m-2}(3)$ (the sixth case of (7.1)). The remaining possibilities for s and t give quotients of $\mathrm{PSp}_{m-2}(3) \ltimes (\mathbb{Z}_3)^{m-2}$ which

also map onto the same group, and therefore give examples belonging to the seventh case of (7.1).

□

We conclude by remarking on a striking aspect of these final cases: the tensor products of certain pairs of irreducible representations of $\mathrm{PSp}_{m-2}(3) \ltimes (\mathbb{Z}_3)^{m-2}$ or $\mathrm{PSp}_{m-2}(3)$ turn out to be irreducible. In particular, the two Weil representations of $\mathrm{PSp}_{m-2}(3)$ have an irreducible tensor product. It would be interesting to find other examples of faithful projective representations which have an irreducible tensor product. We are aware of a number of "sporadic" examples but only two other infinite families, one arising from square Young diagrams in the representation theorem of A_{n^2} and one from Weil representations of unitary groups over the field with two elements.

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